

# Double-critical graph conjecture for claw-free graphs

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## Abstract

A connected graph  $G$  with chromatic number  $t$  is *double-critical* if  $G - x - y$  is  $(t - 2)$ -colorable for each edge  $xy \in E(G)$ . The complete graphs are the only known examples of double-critical graphs. A long-standing conjecture of Erdős and Lovász from 1966, which is referred to as the *Double-critical Graph Conjecture*, states that there are no other double-critical graphs, i.e., if a graph  $G$  with chromatic number  $t$  is double-critical, then  $G = K_t$ . This has been verified for  $t \leq 5$ , but remains open for  $t \geq 6$ . In this paper, we first prove that if  $G$  is a non-complete double-critical graph with chromatic number  $t \geq 6$ , then no vertex of degree  $t + 1$  is adjacent to a vertex of degree  $t + 1$ ,  $t + 2$  or  $t + 3$  in  $G$ . We then use this result to show that the Double-critical Graph Conjecture is true for double-critical graphs  $G$  with chromatic number  $t \leq 8$  if  $G$  is claw-free.

**Keywords:** graph coloring, double-critical graphs, claw-free graphs

## 1 Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph  $G$ , we will use  $V(G)$  to denote the vertex set,  $E(G)$  the edge set,  $\alpha(G)$  the independence number,  $\omega(G)$  the clique number, and  $\chi(G)$  the chromatic number of  $G$ , respectively. A graph  $G$  is *claw-free* if  $G$  does not contain  $K_{1,3}$  as an induced subgraph. Throughout this paper, a proper vertex coloring of a graph  $G$  with  $k$  colors is called a *k-coloring* of  $G$ , where  $k \geq 0$  is an integer. In 1966, the following conjecture of Lovász was published by Erdős [6], the so-called Erdős-Lovász Tihany Conjecture.

**Conjecture 1.1** For any integers  $s, t \geq 2$  and any graph  $G$  with  $\omega(G) < \chi(G) = s + t - 1$ , there exist disjoint subgraphs  $G_1$  and  $G_2$  of  $G$  such that  $\chi(G_1) \geq s$  and  $\chi(G_2) \geq t$ .

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Conjecture 1.1 is hard. To date, Conjecture 1.1 has been shown to be true only for values of  $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$ . The case  $(2, 2)$  is trivial. The case  $(2, 3)$  was shown by Brown and Jung in 1969 [3]. Mozhan [10] and Stiebitz [13] each independently showed the case  $(2, 4)$  in 1987. The cases  $(3, 3)$ ,  $(3, 4)$ , and  $(3, 5)$  were also settled by Stiebitz in 1987 [14]. Recent work on the Erdős-Lovász Tihany Conjecture has focused on proving the conjecture for certain classes of graphs. Kostochka and Stiebitz [9] showed the conjecture holds for line graphs. Balogh, Kostochka, Prince, and Stiebitz [2] then showed that the conjecture holds for all quasi-line graphs and all graphs  $G$  with  $\alpha(G) = 2$ . More recently, Chudnovsky, Fradkin, and Plumettaz [5] proved the following slight weakening of Conjecture 1.1 for claw-free graphs, the proof of which is long and relies heavily on the structure theorem for claw-free graphs developed by Chudnovsky and Seymour [4].

**Theorem 1.2** Let  $G$  be a claw-free graph with  $\chi(G) > \omega(G)$ . Then there exists a clique  $K$  with  $|V(K)| \leq 5$  such that  $\chi(G \setminus V(K)) > \chi(G) - |V(K)|$ .

The most recent result related to the Erdős-Lovász Tihany Conjecture is due to Stiebitz [15], who showed that for integers  $s, t \geq 2$ , any graph  $G$  with  $\omega(G) < \chi(G) = s + t - 1$  contains disjoint subgraphs  $G_1$  and  $G_2$  of  $G$  with either  $\chi(G_1) \geq s$  and  $\text{col}(G_2) \geq t$ , or  $\text{col}(G_1) \geq s$  and  $\chi(G_2) \geq t$ , where  $\text{col}(H)$  denotes the coloring number of a graph  $H$ .

If we restrict  $s = 2$  in Conjecture 1.1, then the Erdős-Lovász Tihany Conjecture states that for any graph  $G$  with  $\chi(G) > \omega(G) \geq 2$ , there exists an edge  $xy \in E(G)$  such that  $\chi(G \setminus \{x, y\}) \geq \chi(G) - 1$ . To prove this special case of Conjecture 1.1, suppose for a contradiction that no such edge exists. Then  $\chi(G \setminus \{x, y\}) = \chi(G) - 2$  for every edge  $xy \in E(G)$ . This motivates the definition of double-critical graphs. A connected graph  $G$  is *double-critical* if for every edge  $xy \in E(G)$ ,  $\chi(G \setminus \{x, y\}) = \chi(G) - 2$ . A graph  $G$  is  *$t$ -chromatic* if  $\chi(G) = t$ . We are now ready to state the following conjecture which is referred to as the *Double-critical Graph Conjecture*, due to Erdős and Lovász [6].

**Conjecture 1.3** Let  $G$  be a double-critical  $t$ -chromatic graph. Then  $G = K_t$ .

Conjecture 1.3 has been settled in the affirmative for  $t \leq 5$  [10, 13], for line graphs [9], and for quasi-line graphs and graphs with independence number two [2]. Representing a weakening of Conjecture 1.3, Kawarabayashi, Pedersen and Toft [8] have shown that any double-critical  $t$ -chromatic graph contains  $K_t$  as a minor for  $t \in \{6, 7\}$ . As a further weakening, Pedersen [11] showed that any double-critical 8-chromatic graph contains  $K_8^-$  as a minor. Albar and Gonçalves [1] later proved that any double-critical 8-chromatic graph contains  $K_8$  as a minor. Their proof is computer-assisted. The present authors [12] gave a computer-free proof of the same result and further showed that any double-critical  $t$ -chromatic graph contains  $K_9$  as a minor for all  $t \geq 9$ . We note here that Theorem 1.2 does not completely settle Conjecture 1.3 for all claw-free graphs. Recently, Huang and Yu [7] proved that the

only double-critical 6-chromatic claw-free graph is  $K_6$ . We prove the following main results in this paper. Theorem 1.4 is a generalization of a result obtained in [8] that no two vertices of degree  $t + 1$  are adjacent in any non-complete double-critical  $t$ -chromatic graph.

**Theorem 1.4** If  $G$  is a non-complete double-critical  $t$ -chromatic graph with  $t \geq 6$ , then for any vertex  $x \in V(G)$  with  $d(x) = t + 1$ , the following hold:

- (i)  $e(\overline{G[N(x)]}) \geq 8$ , where  $e(H)$  denotes the number of edges in a graph  $H$ ; and
- (ii) for any  $y \in N(x)$ ,  $d(y) \geq t + 4$ . In particular,  $d(y) = t + 4$  either when  $|N(x) \cap N(y)| = t$  and  $\overline{G[N(x)]}$  is isomorphic to  $C_8 \cup \overline{K_{t-7}}$  or when  $|N(x) \cap N(y)| < t$  and  $\overline{G[N(x)]}$  is isomorphic to  $C_8 \cup \overline{K_{t-7}}$  or  $C_5 \cup C_5 \cup \overline{K_{t-9}}$ .

Corollary 1.5 below follows immediately from Theorem 1.4.

**Corollary 1.5** If  $G$  is a non-complete double-critical  $t$ -chromatic graph with  $t \geq 6$ , then no vertex of degree  $t + 1$  is adjacent to a vertex of degree  $t + 1$ ,  $t + 2$  or  $t + 3$  in  $G$ .

We then use Corollary 1.5 to prove the following main result.

**Theorem 1.6** Let  $G$  be a double-critical  $t$ -chromatic graph with  $t \in \{6, 7, 8\}$ . If  $G$  is claw-free, then  $G = K_t$ .

The rest of this paper is organized as follows. In Section 2, we first list some known properties of non-complete double-critical graphs obtained in [8] and then establish a few new ones. In particular, Lemma 2.4 turns out to be very useful. We prove our main results in Section 3.

We need to introduce more notation. Let  $G$  be a graph. For a vertex  $x \in V(G)$ , we will use  $N(x)$  to denote the set of vertices in  $G$  which are adjacent to  $x$ . We define  $N[x] = N(x) \cup \{x\}$  and  $d(x) = |N(x)|$ . Given vertex sets  $A, B \subseteq V(G)$ , we say that  $A$  is *complete* (resp. *anti-complete*) to  $B$  if for every  $a \in A$  and every  $b \in B$ ,  $ab \in E(G)$  (resp.  $ab \notin E(G)$ ). We denote by  $B \setminus A$  the set  $B - A$ ,  $e_G(A, B)$  the number of edges between  $A$  and  $B$  in  $G$ ,  $G[A]$  the subgraph of  $G$  induced on  $A$ , and  $G \setminus A$  the subgraph of  $G$  induced on  $V(G) \setminus A$ . If  $A = \{a\}$ , we simply write  $B \setminus a$ ,  $e_G(a, B)$ , and  $G \setminus a$ , respectively. The complement of a graph  $G$  is denoted by  $\overline{G}$  and a cycle with  $k \geq 3$  vertices is denoted by  $C_k$ . Given two graphs  $G$  and  $H$ , the *union* of  $G$  and  $H$ , denoted  $G \cup H$ , is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ .

## 2 Preliminaries

The following is a summary of the basic properties of non-complete double-critical graphs shown by Kawarabayashi, Pedersen and Toft in [8].

**Proposition 2.1** If  $G$  is a non-complete double-critical  $t$ -chromatic graph, then all of the following are true.

- (a)  $G$  does not contain  $K_{t-1}$  as a subgraph.
- (b) for all edges  $xy$ , every  $(t-2)$ -coloring  $c : V(G) \setminus \{x, y\} \rightarrow \{1, 2, \dots, t-2\}$  of  $G \setminus \{x, y\}$ , and any non-empty sequence  $j_1, j_2, \dots, j_i$  of  $i$  different colors from  $\{1, 2, \dots, t-2\}$ , there is a path of order  $i+2$  with vertices  $x, v_1, v_2, \dots, v_i, y$  in order such that  $c(v_k) = j_k$  for all  $k \in \{1, 2, \dots, i\}$ .
- (c) for any edge  $xy \in E(G)$ ,  $x$  and  $y$  have at least one common neighbor in every color class of any  $(t-2)$ -coloring of  $G \setminus \{x, y\}$ . In particular, every edge  $xy \in E(G)$  belongs to at least  $t-2$  triangles.
- (d) there exists at least one edge  $xy \in E(G)$  such that  $x$  and  $y$  share a common non-neighbor in  $G$ .
- (e) for any edge  $xy \in E(G)$ , the subgraph of  $G$  induced by  $N(x) \setminus N[y]$  contains no isolated vertices. In particular, no vertex can have degree one in  $\overline{G[N(x)]}$ .
- (f)  $\delta(G) \geq t+1$ .
- (g) for any vertex  $x \in V(G)$ ,  $\alpha(G[N(x)]) \leq d(x) - t + 1$ .
- (h) for any vertex  $x$  with at least one non-neighbor in  $G$ ,  $\chi(G[N(x)]) \leq t-3$ .
- (i) for any  $x \in V(G)$  with  $d(x) = t+1$ ,  $\overline{G[N(x)]}$  consists of isolated vertices and cycles of length at least five.
- (j) no two vertices of degree  $t+1$  are adjacent in  $G$ .

We next establish some new properties of non-complete double-critical graphs.

**Lemma 2.2** Let  $G$  be a double-critical  $t$ -chromatic graph and let  $x \in V(G)$ . If  $d(x) = |V(G)| - 1$ , then  $G \setminus x$  is a double-critical  $(t-1)$ -chromatic graph.

**Proof.** Let  $uv$  be any edge of  $G \setminus x$ . Clearly,  $\chi(G \setminus x) = t-1$ . Since  $G$  is double-critical,  $\chi(G \setminus \{u, v\}) = t-2$  and so  $\chi(G \setminus \{u, v, x\}) = t-3$  because  $x$  is adjacent to all the other vertices in  $G \setminus \{u, v\}$ . Hence  $G \setminus x$  is double-critical and  $(t-1)$ -chromatic. ■

**Lemma 2.3** If  $G$  is a non-complete double-critical  $t$ -chromatic graph, then for any  $x \in V(G)$  with at least one non-neighbor in  $G$ ,  $\chi(G \setminus N[x]) \geq 3$ . In particular,  $G \setminus N[x]$  must contain an odd cycle, and so  $d(x) \leq |V(G)| - 4$ .

**Proof.** Let  $x$  be any vertex in  $G$  with  $d(x) < |V(G)| - 1$  and let  $H = G \setminus N[x]$ . Suppose that  $\chi(H) \leq 2$ . Since  $d(x) < |V(G)| - 1$ ,  $H$  contains at least one vertex. Let  $y \in V(H)$  be adjacent to a vertex  $z \in N(x)$ . This is possible because  $G$  is connected. If  $H$  has no edge,

then  $G \setminus (V(H) \cup \{z\})$  has a  $(t-2)$ -coloring  $c$ , which can be extended to a  $(t-1)$ -coloring of  $G$  by assigning all vertices in  $V(H)$  the color  $c(x)$  and assigning a new color to the vertex  $z$ , a contradiction. Thus  $H$  must contain at least one edge and so  $\chi(H) = 2$ . Let  $(A, B)$  be a bipartition of  $H$ . Now  $G \setminus H$  has a  $(t-2)$ -coloring  $c^*$ , which again can be extended to a  $(t-1)$ -coloring of  $G$  by assigning all vertices in  $A$  the color  $c^*(x)$  and all vertices in  $B$  the same new color, a contradiction. This proves that  $\chi(H) \geq 3$ , and so  $H$  must contain an odd cycle. Therefore  $d(x) \leq |V(G)| - 4$ .  $\blacksquare$

**Lemma 2.4** Let  $G$  be a double-critical  $t$ -chromatic graph. For any  $xy \in E(G)$ , let  $c$  be any  $(t-2)$ -coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \dots, V_{t-2}$ . Then

- (i) for any  $i, j \in \{1, 2, \dots, t-2\}$  with  $i \neq j$ , if  $N(x) \cap N(y) \cap V_i$  is anti-complete to  $N(x) \cap V_j$ , then there exists at least one edge between  $(N(y) \setminus N(x)) \cap V_i$  and  $N(x) \cap V_j$  in  $G$ . In particular,  $(N(y) \setminus N(x)) \cap V_i \neq \emptyset$ .
- (ii) if  $d(x) = t+1$  and  $y$  belongs to a cycle of length  $k \geq 5$  in  $\overline{G[N(x)]}$ , then  $d(y) \geq t + e(\overline{G[N(x)]}) - 4$  if  $k \geq 7$ ;  $d(y) \geq \max\{t+2, t + e(\overline{G[N(x)]}) - 5\}$  if  $k = 6$ ; and  $d(y) \geq \max\{t+2, t + e(\overline{G[N(x)]}) - 6\}$  if  $k = 5$ .

**Proof.** Let  $G, x, y, c$  be as given in the statement. For any  $i, j \in \{1, 2, \dots, t-2\}$  with  $i \neq j$ , by Proposition 2.1(c), there must exist a path  $x, u_j, u_i, y$  in  $G$  such that  $c(u_j) = j$  and  $c(u_i) = i$ . Clearly,  $u_j u_i \in E(G)$  and  $u_j \in N(x) \cap V_j$ . Since  $N(x) \cap N(y) \cap V_i$  is anti-complete to  $N(x) \cap V_j$ , we see that  $u_i \in (N(y) \setminus N(x)) \cap V_i$ . This proves (i).

Next assume that  $d(x) = t+1$  and that  $y$  belongs to a cycle, say  $C_k$ , of  $\overline{G[N(x)]}$ , where  $k \geq 5$ . By Proposition 2.1(j),  $d(y) \geq t+2$  and by Proposition 2.1(i),  $\overline{G[N(x)]}$  consists of isolated vertices and cycles of length at least five. Clearly,  $|N(x) \cap N(y)| = t-2$ . By Proposition 2.1(c), we may assume that  $V_i \cap (N(x) \cap N(y)) = \{v_i\}$  for all  $i \in \{1, \dots, t-2\}$ . Let  $\{a, b\} = N(x) \setminus N[y]$ . By Proposition 2.1(g),  $ab \in E(G)$ . We may further assume that  $a \in V_1$  and  $b \in V_2$ . Then  $v_1 a y b v_2$  forms a path on five vertices of  $C_k$ . If  $k \geq 6$ , then  $v_1 v_2 \in E(G)$  and each of  $v_1, v_2$  has precisely one non-neighbor in  $\{v_3, v_4, \dots, v_{t-2}\}$ . We may assume that  $v_1 v_3 \notin E(G)$  and  $v_2 v_n \notin E(G)$ , where  $n = 3$  if  $k = 6$  and  $n = 4$  if  $k \geq 7$ . For any  $i, j \in \{3, 4, \dots, t-2\}$  with  $i \neq j$ , if  $v_i v_j \notin E(G)$ , by Lemma 2.4(i), there exists  $v'_j \in V_j \setminus v_j$  such that  $v'_j y \in E(G)$ . By symmetry, there exists  $v'_i \in V_i \setminus v_i$  such that  $v'_i y \in E(G)$ . Thus for any cycle  $C$  of  $\overline{G[N(x)]} \setminus V(C_k)$ , we see that  $C$  must contain  $|V(C)|$  vertices of  $\{v_3, v_4, \dots, v_{t-2}\} \setminus V(C_k)$  and thus  $y$  must be adjacent to a vertex of  $V_m \setminus v_m$  in  $G$  if  $V_m$  contains a vertex of  $C$ , where  $m \in \{3, 4, \dots, t-2\}$ .

Assume that  $k = 5$ . Then  $d(y) = |N(x) \cap N(y)| + |\{x\}| + e(\overline{G[N(x)]} \setminus V(C_k)) \geq (t-2) + 1 + (e(\overline{G[N(x)]}) - 5) = t + e(\overline{G[N(x)]}) - 6$ . Next assume that  $k = 6$ . Then  $v_n = v_3$ . Since  $N(x) \cap V_3$  is anti-complete to  $\{v_1, v_2\}$ , by Lemma 2.4(i) again,  $y$  has at least one neighbor in each of

$V_1 \setminus v_1$  and  $V_2 \setminus v_2$ . Then  $d(y) \geq (t-2) + 3 + (e(\overline{G[N(x)]}) - 6) = t + e(\overline{G[N(x)]}) - 5$ . Finally assume that  $k \geq 7$ . Then  $v_n = v_4$ . Similar to the argument for the case  $k = 6$ ,  $y$  has at least one neighbor in each of  $V_1 \setminus v_1$  and  $V_2 \setminus v_2$ . As observed earlier, for any  $i, j \in \{3, 4, \dots, t-2\}$  with  $i \neq j$  and  $v_i v_j \notin E(G)$ ,  $y$  has at least one neighbor in each of  $V_i \setminus v_i$  and  $V_j \setminus v_j$ . Hence  $d(y) \geq (t-2) + 1 + (k-3) + (e(\overline{G[N(x)]}) - k) = t + e(\overline{G[N(x)]}) - 4$ , as desired. This completes the proof of (ii). ■

### 3 Proofs of Main Results

In this section, we prove our main results, namely, Theorem 1.4 and Theorem 1.6. We first prove Theorem 1.4.

**Proof of Theorem 1.4.** Let  $G$  and  $x$  be as given in the statement. Let  $H = G[N(x)]$ . Then  $|V(H)| = t + 1$ . By Proposition 2.1(g) and Proposition 2.1(h) applied to the vertex  $x$ ,  $\alpha(H) \leq 2$  and  $\chi(H) \leq t - 3$ . Let  $c^*$  be any  $(t-3)$ -coloring of  $H$ . Then each color class of  $c^*$  contains at most two vertices. Since  $|V(H)| = t + 1$ , we see that at least four color classes of  $c^*$  must each contain two vertices. By Proposition 2.1(e),  $H$  has at least eight vertices of degree two in  $\overline{H}$  and so  $e(\overline{H}) \geq 8$ . This proves (i).

To prove (ii), let  $y \in N(x)$ . Since  $d(x) = t + 1$ , by Proposition 2.1 2.1(i), either  $|N(x) \cap N(y)| = t$  or  $|N(x) \cap N(y)| = t - 2$ . Assume that  $|N(x) \cap N(y)| = t - 2$ . By Proposition 2.1(c),  $y$  belongs to a cycle of length at least 5 in  $\overline{H}$ . By (i),  $e(\overline{H}) \geq 8$ . By Lemma 2.4(ii),  $d(y) \geq t + 4$ . In particular,  $d(y) = t + 4$  when  $\overline{H}$  is isomorphic to  $C_8 \cup \overline{K_{t-7}}$  or  $C_5 \cup C_5 \cup \overline{K_{t-9}}$ . So we may assume that  $|N(x) \cap N(y)| = t$ . Let  $c$  be any  $(t-2)$ -coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \dots, V_{t-2}$ . Since  $\alpha(H) \leq 2$ , we may further assume that  $|N(x) \cap V_1| = |N(x) \cap V_2| = 2$  and  $N(x) \cap V_i = \{v_i\}$  for all  $i \in \{3, 4, \dots, t-2\}$ . By (i),  $e(\overline{H}) \geq 8$  and so by Proposition 2.1(i) applied to the vertex  $x$ , there must exist at least four vertices in  $\{v_3, v_4, \dots, v_{t-2}\}$ , say  $v_3, v_4, v_5, v_6$ , such that  $\overline{H}[\{v_3, v_4, v_5, v_6\}]$  has at least two edges. By Lemma 2.4(i),  $y$  must be adjacent to at least one vertex of  $V_j \setminus v_j$  in  $G$  for at least three values of  $j$ , where  $j \in \{3, 4, 5, 6\}$ . Therefore  $|N(y) \setminus N[x]| \geq 3$  and so  $d(y) = |N[x] \cap N(y)| + |N(y) \setminus N[x]| \geq t + 1 + 3 = t + 4$ . In particular,  $d(y) = t + 4$  when  $\overline{H}$  is isomorphic to  $C_8 \cup \overline{K_{t-7}}$ .

This completes the proof of Theorem 1.4. ■

Before we prove Theorem 1.6, we need Lemma 3.1 below on the maximum degree of double-critical claw-free graphs. We also include here Lemma 3.2, which will not be used in the proof of Theorem 1.6.

**Lemma 3.1** Let  $G$  be a double-critical  $t$ -chromatic graph with  $t \geq 6$ . If  $G$  is claw-free, then for any  $x \in V(G)$ ,  $d(x) \leq 2t - 4$ . Furthermore, if  $d(x) < |V(G)| - 1$ , then  $d(x) \leq 2t - 6$ .

**Proof.** Let  $x \in V(G)$  be a vertex of maximum degree in  $G$  and let  $y \in N(x)$ . Then  $G \setminus \{x, y\}$  has at least one edge because  $G \setminus \{x, y\}$  is  $(t-2)$ -chromatic. Let  $uv$  be any edge of  $G \setminus \{x, y\}$  and let  $c$  be any  $(t-2)$ -coloring of  $G \setminus \{u, v\}$  with color classes  $V_1, V_2, \dots, V_{t-2}$ . We may assume that  $x \in V_{t-2}$ . Since  $G$  is claw-free,  $x$  can have at most two neighbors in each of  $V_1, \dots, V_{t-3}$ . Additionally,  $x$  may be adjacent to  $u$  and  $v$  in  $G$ . Therefore  $d(x) \leq 2t-4$ . If  $d(x) < |V(G)|-1$ , then  $\chi(N(x)) \leq t-3$  by Proposition 2.1(h). Since  $G$  is claw-free, each color class in any  $(t-3)$ -coloring of  $G[N(x)]$  can contain at most two vertices, and so  $d(x) \leq 2t-6$ . ■

**Lemma 3.2** Let  $G$  be a double-critical  $t$ -chromatic graph with  $t \geq 6$ . If  $G$  is claw-free, then for any  $x \in V(G)$ ,  $G[N(x)]$  is  $(2t-1-d(x))$ -connected.

**Proof.** Let  $x \in V(G)$  be any vertex and let  $S$  be a minimal separating set of  $G[N(x)]$ . Since  $G$  is claw-free,  $G[N(x)] \setminus S$  has two components, say  $C_1$  and  $C_2$ , both of which must be cliques. Since  $\delta(G[N(x)]) \geq t-2$  by Proposition 2.1(c), we see that  $|C_i \cup S| \geq t-1$  for all  $i \in \{1, 2\}$ . Then  $d(x) = |C_1| + |C_2| + |S| = |C_1 \cup S| + |C_2 \cup S| - |S| \geq 2t-2-|S|$  and so  $|S| \geq 2t-2-d(x)$ .

Suppose that  $|S| = 2t-2-d(x)$ . Then  $|C_1 \cup S| = |C_2 \cup S| = t-1$ . Since  $\delta(G[N(x)]) \geq t-2$ , any vertex  $v \in C_i$  is complete to  $S \cup (C_i \setminus v)$  for all  $i \in \{1, 2\}$ . Hence,  $S$  is complete to  $C_1 \cup C_2$ . Let  $y \in C_1$  and let  $c$  be any  $(t-2)$ -coloring of  $G \setminus \{x, y\}$ . Then  $|N(x) \cap N(y)| = |S \cup (C_1 \setminus y)| = t-2$ . By Proposition 2.1(c), every vertex of  $S \cup (C_1 \setminus y)$  must be assigned a distinct color by  $c$ . Since  $C_2$  is complete to  $S$  and  $C_2$  is a clique, every vertex of  $C_2 \cup S$  must then be assigned a distinct color by  $c$  as well. Thus  $|C_2 \cup S| \leq t-2$ , contradicting the fact that  $|C_2 \cup S| = t-1$ . ■

We are now ready to prove Theorem 1.6.

**Proof of Theorem 1.6.** Let  $G$  and  $t$  be as given in the statement. Suppose that  $G \neq K_t$ . We choose such a counterexample  $G$  with  $t$  minimum. If  $t \in \{6, 7\}$ , then by Proposition 2.1(d), there exists an edge  $xy \in E(G)$  such that  $x$  and  $y$  have a common non-neighbor. By Proposition 2.1(f) and Lemma 3.1,  $t+1 \leq d(x) \leq 2t-6$  and  $t+1 \leq d(y) \leq 2t-6$ . Thus  $t=7$  and  $d(x)=d(y)=8$ , which contradicts Proposition 2.1(j). This proves that  $t=8$ . We next claim that

(1)  $G$  is 10-regular.

**Proof.** By Lemma 2.2 and the minimality of  $t$ ,  $\Delta(G) \leq |G|-2$ . By Proposition 2.1(f) and Lemma 3.1, we see that  $9 \leq d(x) \leq 10$  for all vertices  $x \in V(G)$ . By Corollary 1.5,  $G$  is 10-regular. ■

(2) For any  $x \in V(G)$ ,  $2 \leq \delta(\overline{G[N(x)]}) \leq \Delta(\overline{G[N(x)]}) \leq 3$ .

**Proof.** Let  $x \in V(G)$ . Then  $x$  has at least one non-neighbor in  $G$ , otherwise  $G = K_{11}$  by (1), a contradiction. By Proposition 2.1(h),  $\chi(G[N(x)]) \leq 5$ . Since  $G$  is claw-free, we see that  $\alpha(G[N(x)]) = 2$  and so every vertex of  $N(x)$  has at least one non-neighbor in  $G[N(x)]$ . By Proposition 2.1(e) and Proposition 2.1(c),  $2 \leq \delta(\overline{G[N(x)]}) \leq \Delta(\overline{G[N(x)]}) \leq 3$ . ■

(3) For any  $x \in V(G)$ ,  $\Delta(\overline{G[N(x)]}) = 3$ , i.e.  $\overline{G[N(x)]}$  is not 2-regular.

**Proof.** Suppose that there exists a vertex  $x \in V(G)$  such that  $\overline{G[N(x)]}$  is 2-regular. Let  $y \in N(x)$  and let  $c$  be any 6-coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \dots, V_6$ . Let  $W = N(x) \cap N(y)$ . Then  $|W| = 7$  because  $\overline{G[N(x)]}$  is 2-regular. By Proposition 2.1(c), we may assume that  $|V_1 \cap W| = 2$  and  $|V_i \cap W| = 1$  for every  $i \in \{2, 3, 4, 5, 6\}$ . Let  $V_1 \cap W = \{v_1, u_1\}$  and  $V_i \cap W = \{v_i\}$  for each  $i \in \{2, 3, 4, 5, 6\}$ . Since  $G$  is claw-free, we may further assume that  $N(x) \cap V_2 = \{v_2, u_2\}$  and  $N(x) \cap V_3 = \{v_3, u_3\}$ . Clearly,  $yu_2, yu_3 \notin E(G)$  and thus  $u_2u_3 \in E(G)$  because  $G$  is claw-free. Since  $\overline{G[N(x)]}$  is 2-regular, we see that  $G[\{v_4, v_5, v_6\}]$  is not a clique. We may assume that  $v_4v_5 \notin E(G)$ . By Lemma 2.4(i),  $N(y) \cap (V_j \setminus \{v_j\}) \neq \emptyset$  for all  $j \in \{4, 5\}$ . Let  $w_4 \in V_4 \setminus v_4$  and  $w_5 \in V_5 \setminus v_5$  be the other two neighbors of  $y$  in  $G$ . Then  $N(y) \setminus N[x] = \{w_4, w_5\}$ . By Lemma 2.4(i),  $v_6$  must be complete to  $\{v_2, v_3, v_4, v_5\}$  in  $G$ . Notice that  $v_6$  is complete to  $\{u_2, u_3\}$  in  $G$  since  $\overline{G[N(x)]}$  is 2-regular. Thus  $v_6$  must be anti-complete to  $\{v_1, u_1\}$  in  $G$  and so  $G[\{x, v_1, u_1, v_6\}]$  is a claw, a contradiction. ■

From now on, we fix an arbitrary vertex  $x \in V(G)$ . By (3), let  $y \in N(x)$  with  $|N(x) \cap N(y)| = 6$ . We choose such a vertex  $y \in N(x)$  so that  $N(x) \setminus N[y]$  contains as many vertices of degree two in  $\overline{G[N(x)]}$  as possible. Let  $c$  be any 6-coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \dots, V_6$ . We may assume that  $V_i \cap N(x) \cap N(y) = \{v_i\}$  for all  $i \in \{1, 2, 3, 4, 5, 6\}$ . Since  $G$  is claw-free, we may further assume that  $N(x) \cap V_j = \{v_j, u_j\}$  for all  $j \in \{1, 2, 3\}$ . Notice that  $y$  is anti-complete to  $\{u_1, u_2, u_3\}$  in  $G$  and so  $G[\{u_1, u_2, u_3\}] = K_3$ . Let  $A = \{u_1, u_2, u_3\}$ ,  $B = \{v_1, v_2, v_3\}$ , and  $C = \{v_4, v_5, v_6\}$ . Let  $H = \overline{G[N(x)]}$ .

(4)  $B$  is not complete to  $C$  in  $G$ .

**Proof.** Suppose that  $B$  is complete to  $C$  in  $G$ . Then  $e_H(C, A) = \sum_{v \in C} d_H(v) - 2e(H[C]) \geq 6 - 2e(H[C])$ . For each  $i \in \{1, 2, 3\}$ ,  $u_i v_i, u_i y \notin E(G)$  and  $d_H(u_i) \leq 3$ . Thus  $e_H(A, C) \leq 3$  and so  $e(H[C]) \geq 2$ . Since  $G$  is claw-free, we have  $e(H[C]) = 2$ . We may assume that  $v_4 v_6 \notin E(H)$ . Then  $v_4 v_6 \in E(G)$  and  $v_5 v_4, v_5 v_6 \notin E(G)$ . Since  $d_H(v_4) \geq 2$  and  $d_H(v_6) \geq 2$ , we may assume that  $u_2 v_4, u_3 v_6 \notin E(G)$ . By the choice of  $y$ ,  $d_H(u_1) = 2$  and  $d_H(v_j) = 2$  for all  $j \in \{4, 5, 6\}$ . Since  $d_H(u_2) = d_H(u_3) = 3$ , by the choice of  $y$  again,  $d_H(v_2) = d_H(v_3) = 3$ . Thus  $G[B] = \overline{K_3}$  and so  $G[\{x\} \cup B]$  is a claw, a contradiction. ■

(5)  $G[C] = K_3$ .



**Proof.** Suppose that  $G[C]$  contains a missing edge, say  $v_4v_5 \notin E(G)$ . By Lemma 2.4(i), there exist  $w_4 \in V_4 \setminus v_4$  and  $w_5 \in V_5 \setminus v_5$  such that  $yw_4, yw_5 \in E(G)$ . By (4), we may assume that  $v_3v_j \notin E(G)$  for some  $j \in \{4, 5, 6\}$ . By Lemma 2.4(i),  $y$  has another neighbor, say  $w_3$ , in  $V_3 \setminus v_3$ . Since  $\{w_3, w_4, w_5\} = N(y) \setminus N[x]$ , by Lemma 2.4(i),  $v_4v_5$  is the only missing edge in  $G[C]$  and  $\{v_1, v_2\}$  is complete to  $C$  in  $G$ . If  $e_H(A, C) = 3$ , then  $d_H(u_i) = 3$  for all  $i \in \{1, 2, 3\}$ . By the choice of  $y$ ,  $d_H(v_3) = 3$ . Notice that for all  $i \in \{4, 5, 6\}$ ,  $e_H(v_i, A \cup \{v_3\}) \geq 1$ , and so by the choice of  $y$ ,  $d_H(v_i) = 3$ . Thus  $e_H(A, C) \geq 5$ , which is impossible. Hence  $e_H(A, C) \leq 2$ . Notice that  $e_H(A, C) = (d_H(v_4) - 1) + (d_H(v_5) - 1) + d_H(v_6) - e_H(v_3, C) \geq 2$ . It follows that  $e_H(A, C) = 2$ ,  $e_H(v_3, C) = 2$  and  $d_H(v_i) = 2$  for all  $i \in \{4, 5, 6\}$ . In particular,  $N(x) \setminus N[y]$  has at most one vertex of degree two in  $H$  but  $N(x) \setminus N[v_3]$  has two vertices of degree two in  $H$ , contradicting the choice of  $y$ . ■

(6)  $A$  is not complete to  $C$  in  $G$  and  $A$  contains at least one vertex of degree three in  $H$ .

**Proof.** By (5),  $G[C] = K_3$ . If  $A$  is complete to  $C$  in  $G$ , then  $G[\{x\} \cup A \cup C] = K_7$ , contrary to Proposition 2.1(a). Thus  $A$  is not complete to  $C$  in  $G$ . We may assume that  $u_3v_4 \notin E(G)$ . Now  $d_H(u_3) = 3$ . ■

(7)  $v_1u_1, v_2u_2$ , and  $v_3u_3$  are the only edges in  $H[A \cup B]$ .

**Proof.** Suppose that  $H[A \cup B]$  has at least four edges. By (5) and (2),  $e_H(C, A \cup B) \geq 6$ . On the other hand,  $e_H(A \cup B, C) = \sum_{v \in A \cup B} d_H(v) - 2e(H[A \cup B]) - 3 \leq 15 - 2e(H[A \cup B])$ . It follows that  $e(H[A \cup B]) = 4$  and  $A \cup B$  contains at most one vertex of degree two in  $H$ . Thus  $e_H(A \cup B, C) \leq 7$  and so at least two vertices of  $C$ , say  $v_4$  and  $v_5$ , are of degree two in  $H$ . Since  $e_H(A, C) \leq 3$  and  $G[C] = K_3$  by (5), we may assume that  $v_4v_3 \notin E(G)$ . If  $d_H(v_3) = 3$ , then since  $d_H(v_4) = 2$  and at most one vertex of  $A \cup B$  has degree two in  $H$ , by the choice of  $y$ , exactly one of  $u_1, u_2, u_3$  has degree two in  $H$ . Then  $e_H(A \cup B, C) = 6$ . Thus  $d_H(v_j) = 2$  for all  $j \in \{4, 5, 6\}$  and by the choice of  $y$ , each vertex of  $B$  is adjacent to at most one vertex of  $C$  in  $G$ . Thus  $e_H(A \cup B, C) \leq 5$ , a contradiction. Hence  $d_H(v_3) = 2$ . Now  $d_H(u_i) = 3$  for all  $i \in \{1, 2, 3\}$  because at most one vertex of  $A \cup B$  has degree two in  $H$ . We see that  $N(x) \setminus N[y]$  has no vertex of degree two in  $H$  but  $N(x) \setminus N[u_3]$  has at least one vertex of degree two in  $H$ , contrary to the choice of  $y$ . ■

By (7), we see that for any  $i \in \{1, 2, 3\}$ ,  $v_iv_j \notin E(G)$  for some  $j \in \{4, 5, 6\}$ . By Lemma 2.4(i), let  $w_i \in V_i \setminus v_i$  be such that  $yw_i \in E(G)$  for all  $i \in \{1, 2, 3\}$ . Let  $D = \{w_1, w_2, w_3\}$ . Then  $N(y) \setminus N[x] = D$  and  $G[D] = K_3$  because  $G$  is claw-free. Clearly,  $D$  is not complete to  $C$  in  $G$ , otherwise  $G[\{y\} \cup D \cup C] = K_7$ , contrary to Proposition 2.1(a). We may assume that  $w_3v_4 \notin E(G)$ . For each  $i \in \{1, 2\}$ ,  $v_iv_3, v_iu_3 \in E(G)$  by (7). Thus  $v_1w_3, v_2w_3 \notin E(G)$  because  $G$  is claw-free. Notice that  $w_3, x, v_1, v_2, v_4 \in N(y)$  and  $w_3$  is anti-complete to  $\{x, v_1, v_2, v_4\}$  in  $G$ . Thus  $\Delta(\overline{G[N(y)]}) \geq 4$ , contrary to (2).

This completes the proof of Theorem 1.6. ■

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